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A NEW METHOD OF SOLVING HEAT-CONDUCTION PROBLEMS FOR BODIES OF REVOLUTION

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ABSTRACT

A method is proposed for solving problems of stationary heat conduction for bodies of revolution. When the boundary conditions on the surface of the body are satisfied, this method makes it possible to apply the well-studied procedures for solving planar problems for harmonic functions. The axisymmetric problem is examined in detail. /151*

Let us assume that the body V is obtained when the planar region D, which is bounded by the contour K, is revolved with respect to the z-axis. We shall employ the cylindrical coordinate system ρ, z, ϑ (Ref. 1, 2), and then the equation of the countour K is a function of the variables ρ, z. The essential feature of the method which is advanced consists of the fact that the body of revolution is assumed to be made of a completely nonuniform material, whose thermal conductivity coefficient λ is a differentiable function of the variables ρ, z, ϑ. The imposition of a definite condition on this function results in the fact that the thermal conductivity equation is considerably simplified, which makes it possible to represent the temperature in terms of harmonic functions of the variables ρ, z. When the boundary conditions on the body surface are fulfilled, this enables us to apply the well-known, effective methods of solving planar problems for a wide class of regions D. In spite of the limitation imposed on the function λ, there is a comparatively large amount of freedom for variation in this function. Within a definite range of changes in the variables ρ, z, ϑ, this makes it possible to have λ approximate a given expression, particularly a constant value, and to obtain in this way the effective solution of the thermal conductivity problem for a body made of a material which is close to being uniform.

When there is a stationary thermal regime and when there are no heat sources in the body, the equation of thermal conductivity in a nonuniform medium may be written as follows (Ref. 1, 2)

$$N \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\lambda\frac{\partial T}{\partial\rho}\right)+\frac{\partial}{\partial z}\left(\lambda\frac{\partial T}{\partial z}\right)+\frac{1}{\rho^2}\frac{\partial}{\partial\vartheta}\left(\lambda\frac{\partial T}{\partial\vartheta}\right)\equiv$$
$$\equiv \lambda\left[\Lambda+\frac{\partial}{\partial\rho}(\ln\rho\lambda)\frac{\partial}{\partial\rho}+\frac{\partial}{\partial z}(\ln\lambda)\frac{\partial}{\partial z}+\frac{1}{\rho^2}\frac{\partial}{\partial\vartheta}(\ln\lambda)\frac{\partial}{\partial\vartheta}\right]T=0, \quad (1)$$

* Numbers in the margin indicate pagination in the original foreign text.

where

$$\Lambda = \frac{\partial^2}{\partial\rho^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\vartheta}. \tag{2}$$

We shall try to determine the temperature in the following form

$$T = \frac{1}{w}\Phi, \tag{3}$$

where w, Φ are the differentiable functions of the variables ρ, z, ϑ, $w \neq 0$. Substituting (3) in (1), we may transform equation (1) into the following form

$$N \equiv \frac{\lambda}{w}\left[\Lambda + \frac{\partial}{\partial\rho}\left(\ln\frac{\rho\lambda}{w^2}\right)\frac{\partial}{\partial\rho} + \frac{\partial}{\partial z}\left(\ln\frac{\lambda}{w^2}\right)\frac{\partial}{\partial z} + \right.$$
$$\left. + \frac{1}{\rho^2}\frac{\partial}{\partial\vartheta}\left(\ln\frac{\lambda}{w^2}\right)\frac{\partial}{\partial\vartheta} + g(w)\right]\Phi = 0, \tag{4}$$

where /152

$$g(w) = -\frac{1}{w}\left\{\Lambda w - \frac{2}{w}\left[\left(\frac{\partial w}{\partial\rho}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 + \frac{1}{\rho^2}\left(\frac{\partial w}{\partial\vartheta}\right)^2\right] + \right.$$
$$\left. + \frac{\partial w}{\partial\rho}\frac{\partial}{\partial\rho}\ln\rho\lambda + \frac{\partial w}{\partial z}\frac{\partial}{\partial z}\ln\lambda + \frac{1}{\rho^2}\frac{\partial w}{\partial\vartheta}\frac{\partial}{\partial\vartheta}\ln\lambda\right\}. \tag{5}$$

Setting

$$\lambda = w^2/\rho, \tag{6}$$

we may transform expressions (4), (5) to the following form

$$N \equiv \frac{\lambda}{w}[\Lambda + g(w)]\Phi = 0, \tag{7}$$

$$g(w) = -\frac{1}{w}\Lambda w. \tag{8}$$

Let us examine the solution of equation (7) for the case when the temperature does not depend on ϑ. Taking expression (2) into account, as well as the fact that the operator Λ does not depend on ϑ here, it is advantageous to require $g(w) = 0$. We then obtain the equations which the functions Φ, w must satisfy from (7), (8):

$$N \equiv \frac{\lambda}{w}\left(\frac{\partial^2\Phi}{\partial\rho^2} + \frac{\partial^2\Phi}{\partial z^2}\right) = 0, \tag{9}$$

$$\frac{\partial^2 w}{\partial\rho^2} + \frac{\partial^2 w}{\partial z^2} = 0. \tag{10}$$

The general solution of equation (9) may be written in the form $\Phi = \phi(\xi) + \overline{\phi(\xi)}$ ($\xi = \rho + iz$), where $\phi(\xi)$ is the function which is analytical in the region D; $\overline{\phi(\xi)}$ is the complex conjugate function. Employing formula (3), we obtain the following for the temperature

$$T = \frac{1}{w}[\varphi(\xi) + \overline{\varphi(\xi)}]. \tag{11}$$

Thus, under the conditions (6), (10) the temperature (11) satisfies the equation (1).

Assuming that (10) is a harmonic function of w, let us determine whether it is possible to have the following expression

$$w^2/\rho \tag{12}$$

approximate the given value of λ. For the general case, this problem requires special study, and in order to have expression (12) approximate the given expression λ it is possible to employ the well-known approximation methods of functions by series of functions which are previously known (Ref. 3 - 5). We shall present examples below, where it is shown that even the simplest functions of w exist, by means of which it is possible to have expression (12) approximate a constant value, with an accuracy which is sufficient for practical applications within certain ranges of changes in the variables ρ, z -- i.e., it is possible to obtain the approximate equality

$$\frac{w^2}{\rho} \approx \lambda_0 = \text{const.} \tag{13}$$

Example 1. We have the function

$$w = \sqrt{a}(\rho + \mu), \tag{14}$$

where a, μ are the arbitrary constants satisfying equation (10). According to formula (6), we obtain

$$\lambda = \frac{a}{\rho}(\rho + \mu)^2 \equiv \lambda(\rho). \tag{15}$$

An analysis of the function (15) shows that its derivative with respect to ρ at the point $\rho = \mu$ equals zero, and that at this point the function (15) has a minimum. Let us require

$$\lambda(\rho) = \lambda_0[1 + \Pi_1(\rho)], \quad |\Pi_1(\rho)| \leqslant \delta \tag{16}$$

for

$$0 < \rho_1 \leqslant \rho \leqslant \rho_2, \quad \rho_1 < \mu < \rho_2, \tag{17}$$

where δ is a sufficiently small positive quantity. In order to fulfill condition (16), it is sufficient to determine the constants a, ρ_1, ρ_2 in such a way that

$$\lambda(\mu) = \lambda_0(1 - \delta), \quad \lambda(\rho_1) = \lambda(\rho_2) = \lambda_0(1 + \delta). \tag{18}$$

Setting /153

$$a = \lambda_0\rho_1(1 + \delta)(\rho_1 + \mu)^{-2} \tag{19}$$

and substituting (15), with allowance for (19), in (18), we satisfy equation

(18) for

$$\rho_1 = \frac{\mu}{1-\delta}(\sqrt{1+\delta} - \sqrt{2\delta})^2, \quad \rho_2 = \rho_1\left(\frac{\sqrt{1+\delta}+\sqrt{2\delta}}{\sqrt{1+\delta}-\sqrt{2\delta}}\right)^2 \tag{20}$$

Let us give the values of ρ_1, ρ_2 in the case of $\delta = 0.05$ and 0.1, i.e., for cases when the maximum deviation of $\lambda(\rho)$ from λ_0 = const is 5 and 10%, respectively, within the range (17)

$$\delta = 0{,}05,\ \rho_1 = 0{,}5285\ \mu,\ \rho_2 = 3{,}58\ \rho_1;\ \delta = 0{,}1,\ \rho_1 = 0{,}4024\ \mu,\ \rho_2 = 6{,}18\ \rho_1. \tag{21}$$

Since real bodies are not absolutely uniform in the majority of cases, we may employ formulas (11) and (14) to study the temperature of the bodies, which are customarily assumed to be uniform. When real bodies are calculated for δ, norms must be established which stipulate the limits (17) and (20), where formula (15) is applicable. In particular, if $\delta = 0.05$ or $\delta = 0.1$ may be assumed, we may employ formulas (11) and (14) to study bodies of revolution, whose region D lies within the limits (21) and $-\infty \leqslant z \leqslant \infty$, where μ is an arbitrary constant, $\mu > 0$.

Example 2. If we change to polar coordinates in the ρ, z plane,

$$\xi = re^{i\psi}, \qquad 0 \leqslant r \leqslant \infty, \qquad -\pi/2 \geqslant \psi \geqslant \pi/2, \tag{22}$$

then the following form

$$w = \frac{1}{2}\sqrt{c}\,[(1-ia)\sqrt{\xi} + (1+ia)\sqrt{\bar{\xi}}] = \sqrt{cr}\left(\cos\frac{\psi}{2} - a\sin\frac{\psi}{2}\right) \tag{23}$$

may be assigned to the solution of (10).

Here c, a are real constants. According to (6), we have

$$\lambda = \frac{c}{\cos\psi}\left(\cos\frac{\psi}{2} - a\sin\frac{\psi}{2}\right)^2 \equiv \lambda(\psi). \tag{24}$$

The function $\lambda(\psi)$ at the point $\psi = \psi_0$, which may be determined by the following equation

$$a = \frac{\operatorname{tg}\psi_0 - \operatorname{tg}(\psi_0/2)}{1 - \operatorname{tg}\psi_0\operatorname{tg}(\psi_0/2)} \tag{25}$$

has a minimum. Therefore, we may require

$$\lambda(\psi) = \lambda_0[1 + \Pi_2(\psi)], \qquad |\Pi_2(\psi)| \leqslant \delta, \tag{26}$$

where

$$\psi_1 \leqslant \psi \leqslant \psi_2, \qquad \psi_1 < \psi_0 < \psi_2, \tag{27}$$

and δ is a sufficiently small, positive quantity. In order that the conditions may be fulfilled, the constants c, ψ_1, ψ_2 must be determined so that

$$\lambda(\psi_0) = \lambda_0(1-\delta), \qquad \lambda(\psi_1) = \lambda(\psi_2) = \lambda_0(1+\delta). \tag{28}$$

We obtain the following formula for the constant c

$$c = \frac{\lambda_0(1-\delta)\cos\psi_0}{(\cos\psi_0/2 - a\sin\psi_0/2)^2}. \tag{29}$$

Let us calculate a, ψ_1, ψ_2 for certain ψ_0 for the case $\delta = 0.05$

$$\begin{array}{llll} \psi_0 = 0, & a = 0, & \psi_1 = -34°, & \psi_2 = 34°, \\ \psi_0 = 30°, & a = 0{,}268, & \psi_1 = -4° & \psi_2 = 56°, \\ \psi_0 = 60°, & a = 0{,}5773, & \psi_1 = 39°, & \psi_2 = 73°30', \\ \psi_0 = 80°, & a = 0{,}841, & \psi_1 = 71°, & \psi_2 = 84°45'. \end{array} \tag{30}$$

Thus, if we assume $\delta = 0.05$, we may then employ formulas (11) and (23) to study the temperatures of uniform bodies of revolution, whose region D lies within the limits (27) indicated by formulas (30) for certain cases.

Example 3. If we change to elliptical coordinates (Ref. 6) in the ρ, z plane, we have

$$\rho = \mu \,\mathrm{sh}^{*}\alpha \sin\beta, \qquad z = \mu \,\mathrm{ch}^{*}\alpha \cos\beta, \qquad 0 \leqslant \alpha < \infty, \qquad 0 \leqslant \beta \leqslant \pi, \tag{31}$$

where μ is the scale constant, and equation (10) assumes the following form

$$\frac{\partial^2 w}{\partial\rho^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{\mu^2(\mathrm{sh}^2\alpha + \sin^2\beta)}\left(\frac{\partial^2 w}{\partial\alpha^2} + \frac{\partial^2 w}{\partial\beta^2}\right) = 0. \tag{32}$$

If we assume the solution of (32) has the form

$$w = \sqrt{\mu\lambda_0 c c_1}\,(1 + a\alpha)(1 + b\beta), \tag{33}$$

where c, c_1, λ_0, a, b are constants, then according to formula (6), with allowance for (31), we shall have

$$\lambda = \lambda_0\lambda_1(\alpha)\lambda_2(\beta), \ \lambda_1(\alpha) = c\,\frac{(1+a\alpha)^2}{\mathrm{sh}\,\alpha}, \ \lambda_2(\beta) = c_1\frac{(1+b\beta)^2}{\sin\beta}. \tag{34}$$

The constants c, a, c_1, b and the boundary values α_1, α_2, β_1, β_2 may be determined so that

$$\lambda_1(\alpha) = 1 + \Pi_1(\alpha), \qquad \lambda_2(\beta) = 1 + \Pi_2(\beta), \tag{35}$$

where $|\Pi_1(\alpha)| \leqslant \delta_1$, $|\Pi_2(\beta)| \leqslant \delta_2$; $\alpha_1 \leqslant \alpha \leqslant \alpha_2$; $\beta_1 \leqslant \beta \leqslant \beta_2$; δ_1, δ_2 are sufficiently small, positive numbers.

Let us perform certain computations.

For the function $\lambda_1(\alpha)$: for $c = 0.0876$, $a = 2.78$, $\alpha_1 = 0.2$, $\alpha_2 = 2.4$ we have $\delta_1 = 0.099$, $\lambda_1(0.2) = 1.052$, $\lambda_1(0.4) = 0.95$, $\lambda_1(0.6) = 0.978$, $\lambda_1(0.8) = 1.032$, $\lambda_1(1) = 1.064$, $\lambda_1(1.2) = 1.094$, $\lambda_1(1.4) = 1.096$, $\lambda_1(1.6) = 1.099$, $\lambda_1(1.8) = 1.07$, $\lambda_1(2) = 1.039$, $\lambda_1(2.2) = 0.992$, $\lambda_1(2.4) = 0.944$, $\min\rho = \rho_1 = \mu\,\mathrm{sh}\,\alpha_1 \sin\beta$, $\max\rho = \rho_2 = \mu\,\mathrm{sh}\,\alpha_2\sin\beta$, $\rho_2 : \rho_1 = 27.1$.

* Translator's Note: sh denotes sinh; ch denotes cosh.

For the function $\lambda_2(\beta)$ at the point $\beta = \beta_1$, which may be determined by the equation

$$b = \frac{\operatorname{ctg} \beta_0}{2 - \beta_0 \operatorname{tg} \beta_0} \tag{36}$$

this function has a minimum. We may determine the constants c_1, β_1, β_2 so that $\lambda_2(\beta_0) = 1 - \delta_2$, $\lambda_2(\beta_1) = \lambda_2(\beta_2) = 1 + \delta_2$, where $\beta_1 < \beta_0 < \beta_2$. We have: $c_1 = (1 - \delta_2) \sin \beta_0 (1 + b\beta_0)^{-2}$, and for $\delta_2 = 0.05$

$$\beta_0 = \pi/2, \quad b = 0, \quad \beta_1 = 1{,}14, \quad \beta_2 = 2{,}02,$$
$$\beta_0 = 1{,}32, \quad b = 0{,}134, \quad c_1 = 0{,}664,$$
$$\beta_1 = 0{,}922, \quad \beta_2 = 1{,}793,$$
$$\beta_0 = 1, \quad b = 0{,}473, \quad c_1 = 0{,}368, \quad \beta_1 = 0{,}63, \quad \beta_2 = 1{,}447,$$
$$\beta_0 = 0{,}6, \quad b = 1{,}3, \quad c_1 = 0{,}169, \quad \beta_1 = 0{,}34, \quad \beta_2 = 0{,}98,$$
$$\beta_0 = 0{,}2, \quad b = 4{,}86, \quad c_1 = 0{,}0486, \quad \beta_1 = 0{,}107, \quad \beta_2 = 0{,}37.$$

These examples show that equality (13) is obtained, within an accuracy which is satisfactory for practical applications, even by means of the simplest functions w [see (14), (23), (33)]. Therefore, if formula (11) is employed, the boundary conditions on the surface of the body V may be satisfied for the region D of any complexity, provided that it lies within the limits (17), (27) or (35). If the temperature on the surface of the body is given in the form of the function q, which is a function of the boundary curve of the region D -- contour K -- then the boundary condition assumes the following form

$$\varphi(\xi) + \overline{\varphi(\xi)} = wq. \tag{37}$$

The boundary value problem (37) may be solved for the multiply-connected region D, for which we must employ the well-known methods of solving the planar problem of Dirichlet [for example, (Ref. 7 - 9)].

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